Chapter 5

Linear Equations

The most important task in technical computing.

I am thinking of two numbers. Their average is 3. What are the numbers? Please remember the first thing that pops into your head. I will get back to this problem in a few pages.

Solving systems of simultaneous linear equations is the most important task in technical computing. It is not only important in its own right, it is also a fundamental, and often hidden, component of other more complicated computational tasks.

The very simplest linear equations involve only one unknown. Solve

$$7x = 21$$

The answer, of course, is

$$x = \frac{21}{7} = 3$$

Now solve

$$ax = b$$

The answer, of course, is

$$x = \frac{b}{a}$$

But what if $a = 0$? Then we have to look at b. If $b \neq 0$ then there is no value of x that satisfies

$$0x = b$$

The solution does not exist. On the other hand, if $b = 0$ then any value of x satisfies

$$0x = 0$$

The solution is not unique. Mathematicians have been thinking about existence and uniqueness for centuries. We will see that these concepts are also important in modern technical computing.

Here is a toy story problem.

> Alice buys three apples, a dozen bananas, and one cantaloupe for $2.36.
> Bob buys a dozen apples and two cantaloupes for $5.26. Carol buys two
> bananas and three cantaloupes for $2.77. How much do single pieces of
> each fruit cost?

Let x_1, x_2, and x_3 denote the unknown price of each fruit. We have three equations in three unknowns.

$$
\begin{aligned}
3x_1 + 12x_2 + \ x_3 &= \ 2.36 \\
12x_1 \qquad\quad + 2x_3 &= \ 5.26 \\
2x_2 + 3x_3 &= \ 2.77
\end{aligned}
$$

Because matrix-vector multiplication has been defined the way it has, these equations can be written

$$
\begin{pmatrix} 3 & 12 & 1 \\ 12 & 0 & 2 \\ 0 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2.36 \\ 5.27 \\ 2.77 \end{pmatrix}
$$

Or, simply

$$Ax = b$$

where A is a given 3-by-3 matrix, b is a given 3-by-1 column vector, and x is a 3-by-1 column vector with unknown elements.

We want to solve this equation. If you know anything about matrices, you know that the equation can be solved using A^{-1}, the *inverse* of A,

$$x = A^{-1}b$$

This is a fine concept theoretically, but not so good computationally We don't really need A^{-1}, we just want to find x.

If you do not know anything about matrices, you might be tempted to divide both sides of the equation by A.

$$x = \frac{b}{A}$$

This is a terrible idea theoretically – you can't divide by matrices – but it is the beginning of a good idea computationally.

To find the solution to a linear system of equations with MATLAB, start by entering the matrix of coefficients.

```
A = [3 12 1; 12 0 2; 0 2 3]
```

Since all the elements of `A` are integers, the matrix is printed with an integer format.

```
A =
     3    12     1
    12     0     2
     0     2     3
```

Next, enter the right hand side as a column vector.

```
b = [2.36 5.26 2.77]'
```

The elements of `b` are not integers, so the default format shows four digits after the decimal point.

```
b =
    2.3600
    5.2600
    2.7700
```

MATLAB has an output format intended for financial calculations, like this fruit price calculation. The command

```
format bank
```

changes the output to show only two digits after the decimal point.

```
b =
    2.36
    5.26
    2.77
```

In MATLAB the solution to the linear system of equations

$$Ax = b$$

is found using the *backslash* operator.

```
x = A\b
```

Think of this as "dividing" both sides of the equation by `A`. The result is

```
x =
    0.29
    0.05
    0.89
```

This gives us the solution to our story problem – apples cost 29 cents each, bananas are a nickel each, and cantaloupes are 89 cents each.

Very rarely, systems of linear equations come in the form

$$xA = b$$

where b and x are row vectors. In this case, the solution is found using the forward slash operator.

```
x = b/A
```

The two operations `A\b` and `b/A` are sometimes called *left* and *right* matrix division. In both cases, the coefficient matrix is in the "denominator". For scalars, left and right division are the same thing. The quantities `7\21` and `21/7` are both equal to 3.

Singular matrix

Let's change our story problem a bit. Assume now that Carol buys six apples and one cantaloupe for \$2.77. The coefficient matrix and right hand side become

```
A =
      3    12     1
     12     0     2
      6     0     1
```

and

```
b =
    2.36
    5.26
    2.77
```

At first glance, this does not look like much of a change. However,

```
x = A\b
```

produces

```
Warning: Matrix is singular to working precision.
x =
  NaN
 -Inf
  Inf
```

`Inf` and `-Inf` stand for plus and minus infinity and result from division of nonzero numbers by zero. `NaN` stands for "Not-a-Number" and results from doing arithmetic involving infinities.

The source of the difficulty is that the new information about Carol's purchase is inconsistent with the earlier information about Alice's and Bob's purchases. We have said that Carol bought exactly half as much fruit as Bob. But she paid 2.77 when half of Bob's payment would have been only 2.63. The third row of `A` is equal to one-half of the second row, but b(3) is not equal to one-half of b(2). For this particular matrix A and vector b, the solution to the linear system of equations $Ax = b$ does not exist.

What if we make Carol's purchase price consistent with Bob's? We leave `A` unchanged and revise `b` with

```
b(3) = 2.63
```

so

```
b =
    2.36
    5.26
    2.63
```

Now we do not have enough information. Our last two equations are scalar multiples of each other.

$$12x_1 + 2x_3 = 5.26$$
$$6x_1 + x_3 = 2.63$$

One possible solution is the solution to the original problem.

```
x =
    0.29
    0.05
    0.89
A*x =
    2.36
    5.26
    2.63
```

But we can pick an arbitrary value for any component of the solution and then use the first equation and one of the last two equations to compute the other components. The result is a solution to all three equations. For example, here is a solution with its third component equal to zero.

```
y = A(1:2,1:2)\b(1:2);
y(3) = 0

  y =
    0.44
    0.09
    0.00
A*y =
    2.36
    5.26
    2.63
```

There are infinitely many more.

For this particular matrix A and vector b, the solution to $Ax = b$ is not unique. The family of possible solutions is generated by the *null vector* of A. This is a nonzero vector z for which

$$Az = 0$$

The general form of the solution is one particular solution, say our vector x, plus any arbitrary parameter times the null vector. For any value of t the vector

$$y = x + tz$$

is a solution to

$$Ay = b$$

In MATLAB

```
z = null(A)
A*z
t = rand
y = x + t*z
A*y
```

You can see that `A*z` is zero and `A*y` is equal to `b`.

"Their average is three."

Let's return to the question that I asked you to consider at the beginning of this chapter. I'm thinking of two numbers. Their average is three. What are the numbers?

What popped into your head? You probably realized that I hadn't given you enough information. But you must have thought of *some* solution. In my experience, the most frequent response is "2 and 4". Let's see what MATLAB responds.

My problem is one linear equation in two unknowns The matrix and right hand side are

```
A = [1/2 1/2]
b = 3
```

We want to solve $Ax = b$. This is now an *underdetermined* system. There are fewer equations than unknowns, so there are infinitely many solutions.

Backslash offers one possible solution

```
x = A\b

x =
     6
     0
```

I bet you didn't think of this solution.

If we try `inv(A)*b` we get an error message because rectangular matrices do not have inverses. But there is something called the *pseudoinverse*. We can try that.

```
x = pinv(A)*b

x =
     3
     3
```

Did that solution occur to you?

These two `x`'s are just two members of the infinite family of solutions. If we wanted MATLAB to find the [2 4]' solution, we would have to pose a problem where the solution is constrained to be a pair of integers, close to each other, but not equal. It is possible to solve such problems, but that would take us too far afield.

My Rules.

This chapter illustrates two fundamental facts about technical computing.

- The hardest quantities to compute are ones that do not exist.

- The next hardest are ones that are not unique.

Recap

```
%% Linear Equations Chapter Recap
% This is an executable program that illustrates the statements
% introduced in the Linear Equations Chapter of "Experiments in MATLAB".
% You can access it with
%
%    linear_recap
%    edit linear_recap
%    publish linear_recap

%% Backslash
    format bank
    A = [3 12 1; 12 0 2; 0 2 3]
    b = [2.36 5.26 2.77]'
    x = A\b

%% Forward slash
    x = b'/A'

%% Inconsistent singular system
    A(3,:) = [6 0 1]
    A\b

%% Consistent singular system
    b(3) = 2.63

%% One particular solution
    x = A(1:2,1:2)\b(1:2);
    x(3) = 0
    A*x
```

```
%% Null vector
    z = null(A)
    A*z

%% General solution
    t = rand    % Arbitrary parameter
    y = x + t*z
    A*y
```

Exercises

5.1 *Two-by-two.* Use backslash to try to solve each of these systems of equations. Indicate if the solution exists, and if it unique.

(a)

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} x = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(c)

$$\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

(d)

$$\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} x = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

5.2 *Three-by-three.* Use backslash to try to solve each of these systems of equations. Indicate if the solution exists, and if it unique.

(a)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{pmatrix} x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

(c)

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 12 & 6 \\ 7 & 8 & 12 \end{pmatrix} x = \begin{pmatrix} 3 \\ 12 \\ 15 \end{pmatrix}$$

(d)

$$\begin{pmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(e)

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(f)

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

5.3 *Null vector*. Find a nonzero solution z to

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} z = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

5.4 Matrix equations. Backslash can be used to solve matrix equations of the form

$$AX = B$$

where B has several colums. Do you recognize the solution to the following equation?

$$\begin{pmatrix} 53 & -52 & 23 \\ -22 & 8 & 38 \\ -7 & 68 & -37 \end{pmatrix} X = \begin{pmatrix} 360 & 0 & 0 \\ 0 & 360 & 0 \\ 0 & 0 & 360 \end{pmatrix}$$

5.5 *More Fruit*. Alice buys four apples, two dozen bananas, and two cantaloupes for \$4.14. Bob buys a dozen apples and two cantaloupes for \$5.26. Carol buys a half dozen bananas and three cantaloupes for \$2.97. How much do single pieces of each fruit cost? (You might want to set `format bank`.)

5.6 *Truss*. Figure 5.1 depicts a plane truss having 13 members (the numbered lines) connecting 8 joints (the numbered circles). The indicated loads, in tons, are applied
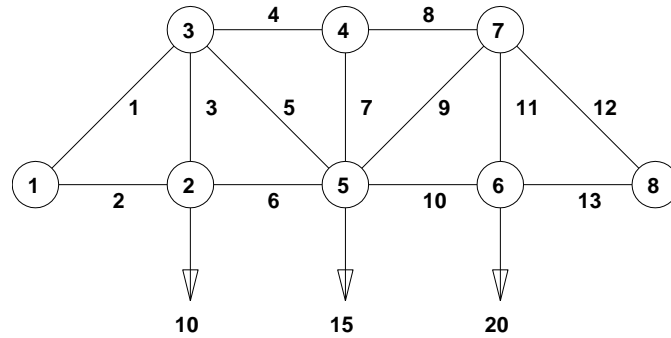
Figure 5.1. *A plane truss.*

at joints 2, 5, and 6, and we want to determine the resulting force on each member of the truss.

For the truss to be in static equilibrium, there must be no net force, horizontally or vertically, at any joint. Thus, we can determine the member forces by equating the horizontal forces to the left and right at each joint, and similarly equating the vertical forces upward and downward at each joint. For the eight joints, this would give 16 equations, which is more than the 13 unknown factors to be determined. For the truss to be statically determinate, that is, for there to be a unique solution, we assume that joint 1 is rigidly fixed both horizontally and vertically and that joint 8 is fixed vertically. Resolving the member forces into horizontal and vertical components and defining $\alpha = 1/\sqrt{2}$, we obtain the following system of equations for the member forces f_i:

$$\begin{aligned}
\text{Joint 2:} \quad & f_2 = f_6, \\
& f_3 = 10; \\
\text{Joint 3:} \quad & \alpha f_1 = f_4 + \alpha f_5, \\
& \alpha f_1 + f_3 + \alpha f_5 = 0; \\
\text{Joint 4:} \quad & f_4 = f_8, \\
& f_7 = 0; \\
\text{Joint 5:} \quad & \alpha f_5 + f_6 = \alpha f_9 + f_{10}, \\
& \alpha f_5 + f_7 + \alpha f_9 = 15; \\
\text{Joint 6:} \quad & f_{10} = f_{13}, \\
& f_{11} = 20; \\
\text{Joint 7:} \quad & f_8 + \alpha f_9 = \alpha f_{12}, \\
& \alpha f_9 + f_{11} + \alpha f_{12} = 0; \\
\text{Joint 8:} \quad & f_{13} + \alpha f_{12} = 0.
\end{aligned}$$

Solve this system of equations to find the vector f of member forces.

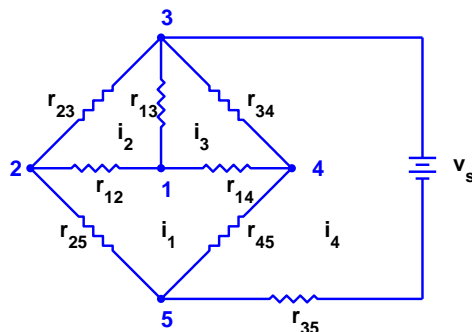5.7 *Circuit.* Figure 5.2 is the circuit diagram for a small network of resistors.

Figure 5.2. *A resistor network.*

There are five nodes, eight resistors, and one constant voltage source. We want to compute the voltage drops between the nodes and the currents around each of the loops.

Several different linear systems of equations can be formed to describe this circuit. Let $v_k, k = 1, \ldots, 4$, denote the voltage difference between each of the first four nodes and node number 5 and let $i_k, k = 1, \ldots, 4$, denote the clockwise current around each of the loops in the diagram. *Ohm's law* says that the voltage drop across a resistor is the resistance times the current. For example, the branch between nodes 1 and 2 gives

$$v_1 - v_2 = r_{12}(i_2 - i_1).$$

Using the *conductance*, which is the reciprocal of the resistance, $g_{kj} = 1/r_{kj}$, Ohm's law becomes

$$i_2 - i_1 = g_{12}(v_1 - v_2).$$

The voltage source is included in the equation

$$v_3 - v_s = r_{35}i_4.$$

Kirchhoff's voltage law says that the sum of the voltage differences around each loop is zero. For example, around loop 1,

$$(v_1 - v_4) + (v_4 - v_5) + (v_5 - v_2) + (v_2 - v_1) = 0.$$

Combining the voltage law with Ohm's law leads to the *loop* equations for the currents:

$$Ri = b.$$

Here i is the current vector,

$$i = \begin{pmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{pmatrix},$$

b is the source voltage vector,

$$b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ v_s \end{pmatrix},$$

and R is the resistance matrix,

$$\begin{pmatrix} r_{25} + r_{12} + r_{14} + r_{45} & -r_{12} & -r_{14} & -r_{45} \\ -r_{12} & r_{23} + r_{12} + r_{13} & -r_{13} & 0 \\ -r_{14} & -r_{13} & r_{14} + r_{13} + r_{34} & -r_{34} \\ -r_{45} & 0 & -r_{34} & r_{35} + r_{45} + r_{34} \end{pmatrix}.$$

Kirchhoff's current law says that the sum of the currents at each node is zero. For example, at node 1,

$$(i_1 - i_2) + (i_2 - i_3) + (i_3 - i_1) = 0.$$

Combining the current law with the conductance version of Ohm's law leads to the *nodal* equations for the voltages:

$$Gv = c.$$

Here v is the voltage vector,

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix},$$

c is the source current vector,

$$c = \begin{pmatrix} 0 \\ 0 \\ g_{35} v_s \\ 0 \end{pmatrix},$$

and G is the conductance matrix,

$$\begin{pmatrix} g_{12} + g_{13} + g_{14} & -g_{12} & -g_{13} & -g_{14} \\ -g_{12} & g_{12} + g_{23} + g_{25} & -g_{23} & 0 \\ -g_{13} & -g_{23} & g_{13} + g_{23} + g_{34} + g_{35} & -g_{34} \\ -g_{14} & 0 & -g_{34} & g_{14} + g_{34} + g_{45} \end{pmatrix}.$$

You can solve the linear system obtained from the loop equations to compute the currents and then use Ohm's law to recover the voltages. Or you can solve the linear system obtained from the node equations to compute the voltages and then use Ohm's law to recover the currents. Your assignment is to verify that these two approaches produce the same results for this circuit. You can choose your own numerical values for the resistances and the voltage source.